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ON THE STRESS-STRAIN STATE NEAR A THREE-DIMENSIONAL CRACK IN A TWO-SHEETED SURFACE*

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A limit scheme of a two-sheeted Riemannian surface is used to illustrate special features encountered in the course of the study of the asymptotic form of the stresses and displacements near the edge of a three-dimensional crack. The fundamental first, second and mixed boundary-value problems are formulated on this surface by analogy with the case of a single plane, and are solved explicitly by quadratures by reducing them to a Riemann boundary-value matrix problem with a constant coefficient. The sheets of the surface are in a generalized plane stress state and have, generally speaking, different stress constants and different thicknesses. A scheme for investigating the stress-strain state of another two-sheeted construction different from the Riemannian surface is elucidated briefly.

A real crystal can naturally be interpreted within the framework of the classical theory of elasticity as a set of elastic interacting planes corresponding to the layers of atoms. Various defects and dislocations /1, 2/ connect the similar surfaces, and it is therefore best to use the methods of the theory of elasticity to multisheeted surfaces when dealing with prescribed types of dislocations.

1. Types of constructions. Let E_1, E_2 be homogeneous, elastic, isotropic infinite thin plates with cuts along the same segment $l_j = [a_j, b_j]$ $(j = 1, 2, \ldots, m)$ of the real x-axis. We shall assume that the plate E_k $(k = 1, 2)$ has a thickness of h_k and is characterized by elastic constants μ_k, $\varkappa_k = (3 - \nu_k)/(1 + \nu_k)$, where μ_k is the shear modulus and ν_k is Poisson's

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ratio. We denote the set of cuts in the plate E_k by L_k, and the upper and lower edges of these cuts by L_k^+ and L_k^- respectively

$$L_k^{\pm} = \bigcup_{j=1}^{m} l_j^{\pm}, \quad l_j \subset E_k$$

Let the plates E_1, E_2 be placed on top of each other so that the j-th cut in the upper plate is situated over the j-th cut of the lower plate E_2 (Fig.1), and the edges of the cuts are joined in a unique manner in one of the following ways.

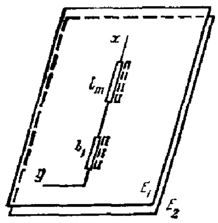

Fig.1

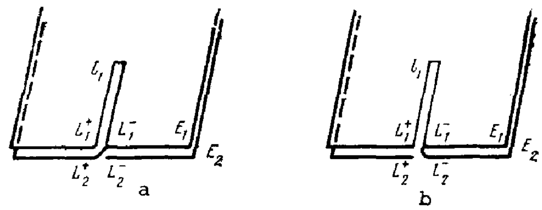

Fig.2

A. The lower edges L_1^- of all cuts in the upper plate are joined to the upper edges L_2^+ of the cut in the lower plate. If we look from the left end of the cut l_1, the joint will look, schematically in the vertical plane without taking into account the thickness of the plates, as in Fig.2a. The resulting system of plates represents a Riemannian surface of the function

$$w = [(z - a_1)(z - b_1)\ldots(z - a_m)(z - b_m)]^{1/2} \tag{1.1}$$

with the edge $L_1^+ \cup L_2^-$. If we mentally join the edges L_1^+ and L_2^-, i.e. make them identical, then we will have a closed Riemannian surface.

B. The lower edges L_1^- of the cuts in the upper plate are joined to the lower edges L_2^- of the cuts in the lower plate (Fig.2b).

We shall assume that: 1) the corresponding cut edges are joined without tension and without any intermediate layers between the edges, by glueing, cross-linking, welding, stamping, etc., 2) the spatial effect of concentration of the stresses along the line of joining is vanishingly small, 3) the plates are in a state of generalized plane stress and interact with each other only through the joined edges of the cuts, 4) the stresses are distributed uniformly at the point at infinity of the plate E_k $(k = 1, 2)$, where the principal stresses $(\sigma_1)_k$ and $(\sigma_2)_k$ act in directions making the angles φ_k and $\varphi_k + \pi/2$, respectively, with the real axis and the rotation at infinity in the plane E_k is equal to ω_k, 5) the stresses and displacement derivatives at the cut ends can become infinite of the order of less than unity, and be continuous at all remaining points of the cuts.

We shall call the problems corresponding to these cases problems A and B, and the constructions themselves the constructions A and B. In all cases the stresses $(\sigma_x, \sigma_y, \tau_{xy})_k$ per unit thickness of the plate and derivatives with respect to x of the displacement components $(u', v')_k$ in the plate E_k will be expressed in terms of two functions $\Phi_k(z)$, $\Psi_k(z)$ $(z = x + iy)$ according to the formulas /3/

$$\begin{gathered}
(\sigma_x + \sigma_y)_k = 4\,\mathrm{Re}\,\Phi_k(z) \\
(\sigma_y - i\tau_{xy})_k = \Phi_k(z) + \Omega_k(\bar z) + (z - \bar z)\overline{\Phi_k'(z)} \\
2\mu_k(u' + iv')_k = \varkappa_k\Phi_k(z) - \Omega_k(\bar z) - (z - \bar z)\overline{\Phi_k'(z)} \\
\Omega_k(z) = \overline{\Phi}_k(z) + z\overline{\Phi}_k'(z) + \overline{\Psi}_k(z), \quad k = 1, 2
\end{gathered} \tag{1.2}$$

The functions are analytic and single-valued in the plane E_k with cuts l_j $(j = 1, 2, \ldots, m)$, and have the following form in the neighbourhood of ∞:

$$\Phi_k(z) = \gamma_k - \frac{P_k}{2\pi(1 + \varkappa_k)}\frac{1}{z} + O(z^{-2}) \tag{1.3}$$

$$\Omega_k(z) = \overline{\gamma}_k + \overline{\gamma}_k' + \frac{\varkappa_k P_k}{2\pi(1 + \varkappa_k)}\frac{1}{z} + O(z^{-2})$$

$$\gamma_k = \frac{1}{4}(\sigma_1 + \sigma_2)_k + \frac{2i\mu_k\omega_k}{1 + \varkappa_k}, \quad \gamma_k' = \frac{1}{2}(\sigma_2 - \sigma_1)\exp(-2i\varphi_k) \tag{1.4}$$

Here $-P_k = -(X_k + iY_k)$ is the principal vector of the forces applied to the set of cuts L_k from the side of E_k per unit thickness of the plate. We shall assume that the numbers P_1, P_2 are known. In some cases, e.g. in the first fundamental problem A, it is sufficient to specify one of these numbers, while the other is found from the boundary conditions and the condition of equilibrium of the whole system of plates E_1, E_2 or from other auxiliary conditions which will be discussed later. At the ends of the cuts the functions Φ_k, Ω_k may become infinite of order less than unity, and at all remaining points of the cuts they will have continuous boundary conditions. Moreover, we shall assume that

$$(z-\bar{z})\,\Phi_k'(z) \to 0 \quad \text{as} \quad z \to t^{\pm},\ k = 1,\ 2 \tag{1.5}$$

at all points of the cuts except the end points.

In all the problems discussed below the above condition is satisfied by virtue of the H-continuity of the given boundary conditions.

2. Problem A. *1°. Formulation of the problem.* We shall assume that at the non-joined edges L_1^+ and L_1^- we know either the normal and shear stresses $(\sigma_y, \tau_{xy})_1^+$ and $(\sigma_y, \tau_{xy})_2^-$ (the first fundamental problem A), or the derivatives in x of the displacement components $(u', v')_1^+$ and $(u', v')_2^-$ (the second fundamental problem A), or that we know the stresses $(\sigma_y, \tau_{xy})_1^+$, at one of the edges, e.g. at L_1^+, and the derivatives of the displacements $(u', v')_2^-$ at the other edge L_2^- (the fundamental mixed problem A). In all cases we assume the given boundary conditions to be H-continuous, and in the second problem we have

$$\int_{l_j} [(u' + iv')_1^+ - (u' + iv')_2^-]\,dx = 0,\ j = 1,\ 2,\ \ldots,\ m \tag{2.1}$$

which expresses the uniqueness of the displacements under the total passage along the cuts over the segment l_j in the plates E_1 and E_2.

Since in this case the system of plates E_1, E_2 with identical edges L_1^+ and L_2^- represents a Riemannian surface of the function (1.1), it follows that the problems formulated above can be regarded as problems on this surface with a three-dimensional cut with the edges L_1^+ and L_2^-, situated in different planes E_1 and E_2.

2°. The boundary value problem for complex potentials. Using relations (1.2) and condition (1.5), we can write the boundary conditions at the edges L_1^+ and L_2^- for all problems A in the following unique form:

$$\nu_1\Phi_1^+(t) + \Omega_1^-(t) = f_1(t),\quad \nu_2\Phi_2^-(t) + \Omega_2^+(t) = f_2(t),\quad t \in L = \bigcup_{j=1}^{m} l_j$$

where $\nu_k = 1,\ f_k(t) = (\sigma_y - i\tau_{xy})_k^{\pm}$ and $\nu_k = -\varkappa_k,\ f_k(t) = -2\mu_k(u' + iv')_k^{\pm}$ provided that the stresses or displacement derivatives are specified on $L_k^{\pm}$. The superscript plus is taken at $k = 1$, and the superscript minus at $k = 2$. The joining of the edges L_1^- and L_2^+ without stretching is described, by virtue of (1.2) and (1.5), by the relations

$$\Phi_1^-(t) + \Omega_1^+(t) = h(\Phi_2^+(t) + \Omega_2^-(t)),\ h = h_2/h_1$$
$$\mu(\varkappa_1\Phi_1^-(t) - \Omega_1^+(t)) = \varkappa_2\Phi_2^+(t) - \Omega_2^-(t),\ \mu = \mu_2/\mu_1,\ t \in L$$

By the same token we obtain the functions $\Phi_k, \Omega_k\ (k = 1, 2)$ from the Riemann boundary value matrix problem

$$\Phi^+(t) = A^{-1}B\Phi^-(t) + A^{-1}f(t),\ t \in L \tag{2.2}$$

$$\Phi(z) = \operatorname{col}\{\Phi_1,\ \Phi_2,\ \Omega_1,\ \Omega_2\},\ f(t) = \operatorname{col}\{f_1,\ f_2,\ 0,\ 0\}$$

$$A = \begin{Vmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -h & 1 & 0 \\ 0 & \varkappa_2 & \mu & 0 \end{Vmatrix},\quad B = \begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & -\nu_2 & 0 & 0 \\ -1 & 0 & 0 & h \\ \varkappa_1\mu & 0 & 0 & 1 \end{Vmatrix} \tag{2.3}$$

The function $\Phi(z)$ may become infinite of order less than unity of the ends of the line L, and in the neighbourhood of ∞ it has, by virtue of (1.3), the form

$$\Phi(z) = G + Hz^{-1} + O(z^{-2}) \tag{2.4}$$

$$G = \operatorname{col}\{\gamma_1, \gamma_2, \bar{\gamma}_1 + \bar{\gamma}_1', \bar{\gamma}_2 + \bar{\gamma}_2'\},$$

$$H = (2\pi)^{-1}\operatorname{col}\{-P_1^0, -P_2^0, \varkappa_1 P_1^0, \varkappa_2 P_2^0\},\quad P_k^0 = P_k/(1 + \varkappa_k),\quad k = 1, 2 \tag{2.5}$$

where $O(z^{-2})$ is a vector function whose every component will be comparable, at large z, with z^{-2}.

3°. *The solution of the problem.* Let λ_k $(k = 1, 2, 3, 4)$ be the eigenvalues of the matrix $A^{-1}B$. In the case of the first and second problem we have

$$\lambda_{1,3} = \pm 1, \quad \lambda_{2,4} = \pm i \left[\frac{\nu_2 (1 + \mu h \varkappa_1)}{\nu_1 (\mu h + \varkappa_2)} \right]^{1/2} \tag{2.6}$$

where $\nu_1 = \nu_2 = 1$ in the first problem, and $\nu_1 = -\varkappa_1$, $\nu_2 = -\varkappa_2$ in the second problem, while in the case of the mixed problem all λ_k will be complex and given by the equation

$$(\mu h + \varkappa_2)\lambda^4 - (2\varkappa_2 - \mu h(\varkappa_1 + \varkappa_2))\lambda^2 + \varkappa_2(1 + \mu h \varkappa_1) = 0 \tag{2.7}$$

We assume that all roots of this equation are different. Then /4/ there exists a non-degenerate matrix S such that the matrix $S^{-1}A^{-1}BS$ will be diagonal with the numbers λ_k $(k = 1, 2, 3, 4)$ serving as its diagonal elements, and the columns of the matrix S will be the eigenvectors of the matrix $A^{-1}B$. The form of the diagonal matrix will be independent of the choice of these vectors. For example, we can take as S

$$S = \left\| \begin{matrix} h\lambda_k (1 + \varkappa_2) \\ 1 + \mu h \varkappa_1 + \lambda_k^2 \nu_1 (\mu h - 1) \\ - h\lambda_k^2 \nu_1 (1 + \varkappa_2) \\ \lambda_k (\varkappa_2 - \mu h \varkappa_1 - \lambda_k^2 \nu_1 (\mu h + \varkappa_2)) \end{matrix} \right\|_{k=1,2,3,4} = \| S_{jk} \| \tag{2.8}$$

The symbol $\| \ldots \|_{k=1,2,3,4}$ denotes here a four-column matrix whose k-th column consists of the elements shown within the bracket. In what follows, we shall also encounter the symbol $\| \ldots \|_4^{k=1}$, which denotes a column vector consisting of the four elements shown within the bracket.

We shall seek the function $\Phi(z)$ in the form

$$\Phi(z) = SF(z) \tag{2.9}$$

where $F(z) = \| F_k \|_4^{k=1}$ is a new unknown piecewise-holomorphic vector function. Then from (2.2) we obtain

$$F_k^+(t) = \lambda_k F_k^-(t) + g_k(t), \; t \in L, \; k = 1, 2, 3, 4 \tag{2.10}$$

where g_k are components of the column vector $\| g_k \|_4^{k=1} = S^{-1}A^{-1}f$. The functions F_k may become infinite of order less than unity at the ends of the line L, and near ∞ they will have, by virtue of (2.4) and (2.9), the form

$$\| F_k \|_4^{k=1} = F(z) = S^{-1}\Phi = S^{-1}G + S^{-1}Hz^{-1} + O(z^{-2}) \tag{2.11}$$

According to /5/ the solutions of problems (2.10) will be the functions, which we shall write in the following unique form in order to facilitate the subsequent calculations:

$$F_k(z) = X_k(z) \left(\frac{1}{2\pi i} \int_L \frac{g_k(t)}{X_k^+(t)} \frac{dt}{t - z} + c_{k0} + c_{k1}z + \ldots + c_{km}z^m \right) \tag{2.12}$$

$$X_k(z) = \left(\prod_{j=1}^m \frac{z - b_j}{z - a_j} \right)^{\alpha_k + i\beta_k} \prod_{j=1}^m \frac{1}{z - b_j} \tag{2.13}$$

$$\alpha_k + i\beta_k = (\ln \lambda_k)/(2\pi i), \; k = 1, 2, 3, 4$$

where $\ln \lambda_k$ are defined in such a manner that $0 \leqslant \operatorname{Im} \ln \lambda_k < 2\pi$, and in the case of multi-valued functions X_k we take the branches which are single-valued in the place with a cut in L and satisfy the conditions $\lim z^m X_k(z) = 1$ as $z \to \infty$. Moreover, since $\lambda_1 = 1$, we must put in the first and second problem

$$X_1(z) = 1, \; c_{11} = c_{12} = \ldots = c_{1m} = 0 \tag{2.14}$$

In the first and second problem we obtain, from (2.11) and (2.12), taking into account (2.14),

$$\left\| \begin{matrix} c_{10} \\ \\ c_{2m} \\ c_{3m} \\ c_{4m} \end{matrix} \right\| = S^{-1}G, \quad \left\| \begin{matrix} -(2\pi i)^{-1} \int\limits_L g_1(t)\,dt \\ \\ c_{2,m-1} + q_2 c_{2m} \\ c_{3,m-1} + q_3 c_{3m} \\ c_{4,m-1} + q_4 c_{4m} \end{matrix} \right\| = S^{-1}H \tag{2.15}$$

$$q_k = (\alpha_k + i\beta_k)(a_1 + a_2 + \ldots + a_m) + (1 - \alpha_k - i\beta_k)(b_1 + b_2 + \ldots + b_m)$$

The form of the first element of the column vector $S^{-1}H$ follows from the boundary conditions and the condition of equilibrium of the construction A. In the mixed problem, (2.11) and (2.12) yield

$$\| c_{km} \|_4^{k=1} = S^{-1}G, \ \| c_{k,\,m-1} + q_k c_{km} \|_4^{k=1} = S^{-1}H \tag{2.16}$$

where q_k are found in the same manner as in the first and second problem.

4°. *Determination of the constants* c_{kj}. If $m = 1$, then relations (2.14)-(2.16) will yield the values of all the constants c_{kj}. When $m > 1$, to determine the remaining constants in the first problem we must require that the displacement increments along the closed curves formed from the cut edges l_j $(j = 1, 2, \ldots, m-1)$ in every plate E_1, E_2 and the curves formed from the segments $[b_j, a_{j+1}]$ $(j = 1, 2, \ldots, m-1)$ situated in E_1 and E_2, be equal to zero. This can be explained by the fact that construction A represents a $(3m-2)$-ply connected region on the Riemannian surface of the algebraic function (1.1) /6/. We have in this region $3m-3$ of the mutually non-homotopic closed curves such that all remaining closed curves can be obtained from them by (one or several) continuous deformation(s) within the boundaries of the region. The curves shown above, along which the displacement increments must be equal to zero, are examples of such non-homotopic curves. From this we obtain, using (1.2), (2.3) and (2.9),

$$\sum_{k=1}^{4} (\varkappa_1 S_{1k} + S_{3k}) \int\limits_{l_j} (F_k^+(t) - F_k^-(t))\,dt = 0 \tag{2.17}$$

$$\sum_{k=1}^{4} (\varkappa_2 S_{2k} + S_{4k}) \int\limits_{l_j} (F_k^+(t) - F_k^-(t))\,dt = 0 \tag{2.18}$$

$$\sum_{k=1}^{4} (\mu(\varkappa_1 S_{1k} - S_{3k}) - \varkappa_2 S_{2k} + S_{4k}) \int\limits_{b_j}^{a_{j+1}} F_k(t)\,dt = 0 \tag{2.19}$$

$$j = 1, 2, \ldots, m-1$$

where S_{jk} are the elements of the matrix S. Replacing F_k in these equations by their values, we obtain a system of $3m-3$ linear algebraic equations for determining the remaining $3m-3$ constants c_{kj} $(k = 2, 3, 4;\ j = 0, 1, \ldots, m-2)$. The unique solvability of this system is proved in the same way as the classical case (/3/, p.442).

In the second problem conditions (2.18) follow from conditions (2.1) and (2.17), therefore they must be replaced by specifying additional $m-1$ conditions. These conditions will be obtained, provided that the difference between the displacements of the points b_j a_{j+1} are known. Then, according to Eqs.(1.2), (2.3) and (2.9) we will have

$$\sum_{k=1}^{4} (\varkappa_1 S_{1k} - S_{3k}) \int\limits_{b_j}^{a_{j+1}} F_k(t)\,dt = 2\mu_1 [u(a_{j+1}) + iv(a_{j+1}) - u(b_j) - iv(b_j)], \tag{2.20}$$

$$j = 1, 2, \ldots, m-1$$

Instead of the difference between the displacements of the points b_j and a_{j+1}, we can also specify the principal vectors of external forces acting on the non-joined edges of the cuts l_j^+ in plate E_1, and l_j^- in plate E_2, together or on one of the edges. Then one of the following conditions must hold $(j = 1, 2, \ldots, m-1)$:

$$\sum_{k=1}^{4} \int\limits_{l_j} (S_{1k} F_k^+(t) + S_{3k} F_k^-(t))\,dt = iQ_{j1} \tag{2.21}$$

$$\sum_{k=1}^{4} \int\limits_{l_j} (S_{2k} F_k^-(t) + S_{4k} F_k^+(t))\,dt = -iQ_{j2} \tag{2.22}$$

$$\sum_{k=1}^{4} \int_{l_j} ((S_{1k} - S_{4k}) F_k^+(t) + (S_{3k} - S_{2k}) F_k^-(t))\, dt = iQ_j \tag{2.23}$$

where Q_{j1}, Q_{j2} and Q_j are the principal vectors of external forces acting, respectively, on the edge l_j^+, in plate E_1, on the edge l_j^- in plate E_2, and on the set of edges l_j^+ in E_1 and l_j^- in E_2. The conditions (2.17) and (2.19) and one of the conditions (2.20)-(2.23) form, at every $j\ (j = 1, 2, \ldots, m-1)$, a uniquely solvable system of $3m-3$ equations for determining $3m-3$ unknown constants c_{kj}, and different conditions can be taken from the group of conditions (2.20)-(2.23) for different j.

In the mixed problem for determining $4m-4$ constants $c_{kj}\ (k = 1, 2, 3, 4;\ j = 0, 1, \ldots, m-2)$, we must take the conditions (2.17)-(2.19) and one of the conditions (2.20)-(2.23) for every j. The conditions form a uniquely solvable system of $4m-4$ equations.

5°. *Methods of specifying the numbers* P_1, P_2. In the first problem it is sufficient to specify one of these numbers. The other number can be found from the relation

$$h_1 P_1 + h_2 P_2 = -\sum_{j=1}^{m} \int_{l_j} (h_1 (\tau_{xy} + i\sigma_y)_1^+ - h_2 (\tau_{xy} + i\sigma_y)_2^-)\, dt$$

expressing the equilibrium of the whole construction A. Both these numbers can also be found if the difference between the displacements of any two tips of the cuts is known, e.g. the displacement between the points a_1 and b_1. Such a situation arises e.g. in the following problem: for the given stresses in L_1^+ and L_2^- to choose the numbers P_1, P_2 so that the displacements of the points a_1 and b_1, differ from each other by a prescribed amount.

In the second problem we should specify both numbers P_1, P_2, or one of them, and in order to determine the other one we should specify the principal vector of external forces applied to one of the edges L_1^+ and L_2^-, or to the set of edges L_1^+, L_2^-. The numbers P_1, P_2 can also be found if the principal vectors of the forces acting at the edges L_1^+ and L_2^- respectively are given.

In the mixed problem both numbers P_1, P_2 must be specified, or one of them and the principal vector of external forces acting on L_2^-. Then the other number can be found from the condition of equilibrium of construction A. Other methods also exist of specifying the numbers P_1, P_2.

6°. *The behaviour of the stresses and displacement derivatives near the cut ends.* According to (2.12), the functions $F_k(z)$ in the first and second problem have the following form /5/ near the point $z = a_j$:

$$F_1(z) = D_{0j} \ln (z - a_j) + O(1),\quad D_{0j} = -g_1(a_j)/2\pi i \tag{2.24}$$

$$F_k(z) = D_{kj} (z - a_j)^{-\alpha_k - i\beta_k} + O(1),\quad k = 2, 3, 4 \tag{2.25}$$

$$D_{kj} = \eta_{kj}(a_j) \left(\frac{1}{2\pi i} \int_L \frac{g_k(t)}{X_k^+(t)} \frac{dt}{t - z} + c_{k0} + c_{k1} a_j + \ldots + c_{km} a_j^m \right) \tag{2.26}$$

$$\eta_{kj}(z) = X_k(z) (z - a_j)^{\alpha_k + i\beta_k}$$

The functions X_k, g_k are given by (2.13), the numbers c_{kj} are defined above, and we mean by $(z - a_j)^{\alpha_k + i\beta_k}$ the branch, single-valued in the plane with a cut along the ray $[a_j, +\infty)$ of the real axis, which takes the value 1 at the upper edge of the cut at $z - a_j = 1$. The integral in (2.26) exists as an improper integral. In the mixed problem all functions F_k, including F_1, have the form (2.25). In the first and second problem we find from (1.2), (2.3), (2.9), (2.24) and (2.25) the following asymptotic representation of the stresses and displacement derivatives near the point $z = a_j$ in plate E_1:

$$(\sigma_x + \sigma_y)_1 = 4 \operatorname{Re} \Big(\sum_{k=2}^{4} S_{1k} D_{kj} \omega_{kj}(z) \Big) + O(\ln r) \tag{2.27}$$

$$\begin{Bmatrix} (\sigma_y - i\tau_{xy})_1 \\ 2\mu_1 (u' + iv')_1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \varkappa_1 \end{Bmatrix} \Big(\sum_{k=2}^{4} S_{1k} D_{kj} \omega_{kj}(z) \Big) + \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Big(\sum_{k=2}^{4} [S_{3k} D_{kj} \omega_{kj}(\bar z) +$$

$$(\alpha_k - i\beta_k) \bar S_{1k} \bar D_{kj} (1 - (z - a_j) / \overline{(z - a_j)})\, \overline{\omega_{kj}(z)}] \Big) + O(\ln r)$$

$$r = |z - a_j|,\quad \omega_{kj}(z) = (z - a_j)^{-\alpha_k - i\beta_k},\quad j = 1, 2, \ldots, m$$

The constants α_k, β_k, S_{jk} and D_{kj} are given by (2.6), (2.8), (2.13) and (2.26), respectively.

In the mixed problem we must take all sums in the above representations over k, from 1 to 4. The constants α_k and β_k are found from (2.13) and (2.7).

In order to obtain the analogous representations near the point $z = b_j$ in the plate E_1, we must replace a_j in (2.24)-(2.27) by b_j, $\alpha_k + i\beta_k$ on $1 - \alpha_k - i\beta_k$, and to obtain the representations near the tips of the cuts in the plate E_2 we must replace $\varkappa_1$ by $\varkappa_2$, μ_1 by μ_2 and the numbers S_{1k} and S_{3k} by S_{2k} and S_{4k} $(k = 1, 2, 3, 4)$, respectively.

The numbers D_{kj} in representations (2.25) and (2.27) play the part of the stress intensity coefficients, and are used in calculating the invariant Γ-integrals /7, 8/.

From the representations (2.27) we see that the stresses and displacement derivatives near the point $z = a_j$ have, in general, not only power-type singularities, but also oscillatory singularities determined by functions of the form $(z - a_j)^{-i\beta_k}$ or, without loss of generality, by a function of the type $[(z - a_j)/(b_j - a_j)]^{-i\beta_k}$. Here, in the case of the first and second problem, the highest-order singularity is determined by the function $(z - a_j)^{-i\beta_4 - 3/4}$. A similar pattern is observed in e.g. (/3/, ch.6) the classical problem of a stamp, or in a mixed problem for a plane with rectilinear cuts. According to relations (2.13) and (2.6),

$$\beta_1 = \beta_3 = 0, \qquad \beta_2 = \beta_4 = -\frac{1}{4\pi} \ln \frac{\nu_2 (1 + \mu h \varkappa_1)}{\nu_1 (\mu h + \varkappa_2)}$$

In particular, if $\mu h = (\nu_1\varkappa_2 - \nu_2) / (\nu_2\varkappa_1 - \nu_1)$, then $\beta_2 = \beta_4 = 0$ and we have no oscillatory-type singularities. Since the expression under the ln sign increases as a function of μh, it follows that its values at $5/3 < \varkappa_1$, $\varkappa_2 < 3$ (these constants are such in the case of real plates) and $\mu h > 0$ lie between $1/3$ and 3. Therefore

$$|\beta_2| = |\beta_4| < (\ln 3)/4\pi \approx 0.0874$$

and the oscillatory-type singularities will manifest themselves appreciably only when

$$|z - a_j| < (b_j - a_j) \exp[-2\pi^2/\ln 3] \approx 1.573 \cdot 10^{-8} (b_j - a_j) \tag{2.28}$$

where $(b_j - a_j)$ is the length of the cut l_j. Clearly, the representations (2.27) occur near the point $z = a_j$ outside a sufficiently small neighbourhood determined by the inequality (2.28).

7°. *A special case.* Let us consider the case when we have a single cut $[a_1, b_1] = [-a, a]$ and $\varkappa_1 = \varkappa_2 = \varkappa$, $\mu_1 = \mu_2$, $h_1 = h_2$, i.e. when the construction A represents a homogeneous Riemannian surface of constant thickness. We can then write the solutions of all problems A in the unique form

$$\Phi(z) = SF(z), \quad \Phi = \begin{Vmatrix} \Phi_1 \\ \Phi_2 \\ \Omega_1 \\ \Omega_2 \end{Vmatrix}, \quad S = \begin{Vmatrix} \lambda_k \\ 1 \\ -\nu\lambda_k^2 \\ -\nu\lambda_k^3 \end{Vmatrix}_{k=1,2,3,4}$$

$$F = \|F_k\|_4^{k=1}, \quad F_k(z) = X_k(z)\left(\frac{1}{2\pi i}\int_{-a}^{a} \frac{g_k(t)}{X_k^+(t)}\frac{dt}{t - z} + c_{k0} + c_{k1}z\right)$$

$$\|c_{k0} + q_k c_{k1}\|_4^{k=1} = S^{-1}H, \quad \|c_{k1}\|_4^{k=1} = S^{-1}G, \quad \|g_k\|_4^{k=1} = S^{-1}A^{-1}f(t)$$

where in the first and mixed problem $\nu = \nu_1 = 1$, in the second problem $\nu = -\varkappa$, in the first and second problem

$$\lambda_{1,3} = \pm 1, \ \lambda_{2,4} = \pm i, \ q_{1,3} = 0, \ q_{2,4} = \pm a/2$$

$$X_1(z) = \frac{1}{z}, \quad X_k(z) = (z - a)^{(k-5)/4}(z + a)^{(1-k)/4}, \quad k = 2, 3, 4$$

in the mixed problem

$$\lambda_k = \varkappa^{1/4} \exp[i\pi(2k - 1)/4], \ q_k = a(5 - 2k + 8i\beta)/4$$

$$X_k(z) = (z - a)^{-i\beta + (2k-9)/8}(z + a)^{i\beta + (1-2k)/8}, \quad \beta = \frac{\ln \varkappa}{8\pi}, \quad k = 1, 2, 3, 4$$

and the column vector G, H, f and matrix A are given by (2.5), (1.4) and (2.3).

Relations (2.27), in which we must put

$$\omega_{kj}(z) = (z - a)^{(k-5)/4} \text{ or } \omega_{kj}(z) = (z - a)^{-i\beta + (2k-9)/8}$$

hold for the stresses and displacement derivatives near the point a. The largest singularity in these representations is determined in the first and second problem by the function $(z-a)^{-3/4}$, and in the mixed problem by the function $(z-a)^{-i\beta-7/8}$.

3. ***Problem B.*** As in problem A, we assume that either the stresses (the first problem B) or the displacements (the second problem B) are specified at the non-joined edges L_1^+ and L_2^+, or the stresses are specified at one edge and the displacements at the other edge (the mixed problem B). In addition we must specify the boundary conditions on the line along which the edges L_1^- and L_2^- are joined (Fig.2b). If it is the displacements that are specified on this line, then problem B separates into two independent problems separately for plates E_1 and E_2, solved in /3/. We shall assume that the external forces $(X+iY)_{\text{ext}}$ are specified on this line. Then we shall obtain the Riemann matrix problem (2.2) with coefficients

$$A=\begin{Vmatrix}\nu_1 & 0 & 0 & 0\\ 0 & \nu_2 & 0 & 0\\ 0 & 0 & -\mu & 1\\ 0 & 0 & 1 & h\end{Vmatrix},\quad B=\begin{Vmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ -\mu\varkappa_1 & \varkappa_2 & 0 & 0\\ -1 & -h & 0 & 0\end{Vmatrix},\quad f(t)=\begin{Vmatrix}f_1\\ f_2\\ 0\\ f_4\end{Vmatrix}$$

$$h=h_2/h_1,\ \mu=\mu_2/\mu_1,\ f_4(t)=-i\ (X+iY)_{\text{ext}}$$

where

$$\nu_k=1,\ f_k(t)=(\sigma_y-i\tau_{xy})_k^+$$

or

$$\nu_k=-\varkappa_k,\ f_k(t)=-2\mu_k(u'+iv')_k^+\ (k=1,2)$$

for determining the complex potentials $\Phi_1, \Phi_2, \Omega_1, \Omega_2$, forming the column vector $\Phi(z)$, depending on whether the stresses or displacements are specified on L_k^+. From then on problem B is solved just like problem A. Moreover, all results obtained for problem A in subsection 3°-6° of Sect.2 hold for problem B with the sole difference, that the eigenvalues λ_k of the matrix $A^{-1}B$ are changed as well as the matrix S whose columns are eigenvectors of the matrix $A^{-1}B$. In the case of the first and second problem B we have

$$\lambda_{1,3}=\pm 1,\ \lambda_{2,4}=\pm i\,[(\mu h\varkappa_1+\varkappa_2)\,/\,(\nu_1\nu_2(\mu h+1))]^{1/2}$$

while in the mixed problem all λ_k are complex and given by the equation

$$\varkappa_2(\mu h+1)\lambda^4+(\mu h(1+\varkappa_1\varkappa_2)-2\varkappa_2)\lambda^2+\mu h\varkappa_1+\varkappa_2=0$$

The matrix S has the form

$$S=\begin{Vmatrix}\varkappa_2-h+\nu_2\lambda_k^2(h+1)\\ 1+\mu\varkappa_1+\nu_1\lambda_k^2(\mu-1)\\ -\nu_1\lambda_k(\varkappa_2-h+\nu_2\lambda_k^2(h+1))\\ -\nu_2\lambda_k(1+\mu\varkappa_1+\nu_1\lambda_k^2(\mu-1))\end{Vmatrix}_{k=1,2,3,4}$$

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